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Mathematics and Soft Minds

Characteristic Functions in Statistics

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A Note [illegible] to the Square of the Circle

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Mathematics and Soft Minds

Under pressure of war needs, American schools are reorganizing their curricula with respect to elementary mathematics. Carefully measured quantities of arithmetic, algebra, geometry, trigonometry are being required of the pre-naval and pre-aeronautic student.

Thousands of young men, finishing four-year college programs deliberately made-up to exclude mathematics, or to include the scantiest amounts of it, on moving to serve their country, are being told to go back and get their algebra and their trigonometry.

Taking their cues from the softness of the times and from the widespread doctrine (largely false and superficial) that options in his studies are a blessing to the student, high school administrations, in too many cases, had reduced to almost vanishing proportions the mathematics required for graduation. The belief, dear to our pioneer forebears, that mental muscles, like physical ones, are hardened by doing hard things, was being flaunted as a teaching entirely against Nature's grain. If budding young America is impelled by his inheritance to football or to swimming, rather than to the mind-bruising athletics of brutal mathematics, let him alone! It is Nature's way!

Has it been left for a world war to bring us to our senses and by virtue of stern *compulsion* to place before us once more the *truth* that *discipline* is a real key to every sort of training?

Whether the faculty to be trained is physical or mental matters not; the *strength* of that faculty is proportioned to the severity of the disciplining process. Every worthwhile training is measurably *against* Nature, rather than wholly *with* Nature.

May this present earth-wide demonstration of the *power* of mathematics in a war crisis carry over to post-war times regenerated conceptions by the American public of the *preeminent* place mathematical science should have in every program of our schools.

S. T. Sanders.

Characteristic Functions in Statistics*

By J. F. KENNEY
University of Wisconsin

1. *Introduction.* The first use of an analytic method substantially equivalent to characteristic functions seems to be due to Lagrange.[1]† Similar functions were then extensively employed by Laplace in his classical treatise, *Théorie Analytique des Probabilités.* Essentially the same analytic instrument as the *fonction generatrice* of Laplace was reintroduced under the name characteristic function by Levy[2] in his book on probability. But it is only within the last decade or so that the power and elegance of the calculus of characteristic functions has been generally realized in theoretical statistics. This technique has been used in many of the recent contributions,[3] relating to distribution functions, which are considered fundamentally important. Some of these papers, however, require for their comprehension a high degree of mathematical sophistication. One of the purposes of the present undertaking is to give an elementary exposition which will facilitate an appreciation of the theory and applications of characteristic functions in mathematical statistics. A second objective may be explained as follows. Certain distribution functions used in sampling theory, originally reached by geometric methods, admit also of analytic formulation. This paper represents an effort to provide the tools by by which such formulation may be accomplished at the level of undergraduate courses and books in mathematics.

Although all intentions toward a formal treatment of the characteristic function are disclaimed, it is hoped that the treatment will be generally interesting and useful to students of mathematical statistics including teachers who may have only a "bowing acquaintance" with this technique.

2. *Distribution functions and expected values.* A continuous variable x is said to have the distribution function $f(x)$ if the frequency of occurence in the range $a \leqq x \leqq b$ is measured by

$$\int_a^b f(x)dx. \tag{1}$$

*On account of its length, this paper is being published in two installments. The second installment will appear in the next issue.—EDITOR.

†Numbers in square brackets refer to the Bibliography at the end of the paper.

It will be understood that $f(x)$ is single-valued and is non-negative for all real values of x. Also it will be assumed that a normalizing constant factor in $f(x)$ is determined so that

$$\int_{-\infty}^{\infty} f(x)dx = 1. \tag{2}$$

If the actual occurrence of the variable is limited to a finite range, $f(x)$ is defined to be identically zero outside that range. Under the condition specified by the second integral, the first integral denotes the probability that x lies in the interval ab. Then $f(x)$ is also called a probability function.

For the discrete case,* the condition corresponding to (2) is

$$\sum_x f(x) = 1, \tag{2a}$$

where the notation $\sum_x$

denotes summation over all admissible values of the discrete variable x.

A variable associated with a probability function in accordance with the above definitions is sometimes called a *variate*.

If $g(x)$ is an arbitrary function of x and $f(x)$ is the distribution function, then the sum or integral

$$\sum_x g(x)f(x), \qquad \int_{-\infty}^{\infty} g(x)f(x)dx \tag{3}$$

defines the ***expected value*** of $g(x)$. It is commonly denoted by $E\{g(x)\}$. This definition is subject, of course, to the condition that the appropriate expression in (3) exists. We are particularly interested in these expressions when $g(x)$ takes certain simple forms. If $g(x) = x^r$, where here and subsequently r is a positive integer, then (3) defines the rth moment ν_r about the origin of x. In particular, $\nu_1 = E(x) =$ mean of x. When $g(x) = (x - \nu_1)^r$, (3) defines the rth moment μ_r about the mean. We write $\mu_2 = \sigma^2$. It is conventional to call σ the ***standard deviation*** and σ^2 the ***variance*** of x. Either σ or σ^2 may be taken as a suitable measure of dispersion, although modern usage favors the latter.

Variates $x_1, x_2, \cdots, x_N$ are said to be ***independent*** in the statistical sense if the function defining their joint distribution can be expressed identically as the product of their separate distribution functions.

*The discrete and continuous cases can both be included under a Stieltjes integral.[4]

Otherwise they are said to be *correlated.* We shall have occasion to use the following propositions which are established in the literature.[5]

Proposition 1. The expected value of an algebraic sum of two or more variates is the same algebraic sum of the expected values of the variates.

Proposition 2. The expected value of a product of independent variates is equal to the product of their expected values.

Proposition 3. If the variates are correlated in pairs the covariance σ_{jk} of x_j and x_k is defined as follows:

$$\begin{aligned}\sigma_{jk} &= E(x_j x_k) - E(x_j)E(x_k)\\ &= \rho_{jk}\sigma_j\sigma_k\end{aligned}$$

where ρ_{jk} is the *coefficient of correlation* between x_j and x_k. If $j=k$, then $\rho_{jj}=1$ and σ_{jj} is the variance of x_j.

3. *Characteristic functions.* When $g(x)=e^{wx}$, where w is an auxiliary variable, the expressions in (3) define the characteristic function of the distribution of x. This function will be denoted by $M(w,x)$. Thus we have

$$M(w,x) = E(e^{wx}) = \sum_x e^{wx}f(x) \quad \text{or} \quad \int_{-\infty}^{\infty} e^{wx}f(x)dx, \tag{4}$$

according as x is a discrete or continuous variable. It is obvious that $M(0,x)=1$.

The above definition may not exist for some functions if w is a real variable. Consider, for example, the Cauchy distribution[6]

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty \leqq x \leqq \infty.$$

Its characteristic function is, from (4)

$$M(w,x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{wx}dx}{1+x^2}.$$

Since the integrand becomes indefinitely large as $x \to \infty$, this integral does not exist for $x=\infty$.

Such difficulties of existence can be avoided by stipulating that w be a pure imaginary. Thus if we let $w=iy$, $i=(-1)^{1/2}$, we have for the discrete case,

$$M(iy,x) = \sum_x e^{iyx}f(x). \tag{5}$$

Since the absolute value of a sum of complex numbers is never greater than the sum of their absolute values whereas the absolute value of a product equals the product of the absolute values, it is clear that

$$\left| \sum_x e^{iyx} f(x) \right| \leqq \sum_x f(x) = 1.$$

The expression defining $M(iy,x)$ in (5) has an interesting geometric interpretation. If we let $\Theta = yx$ represent a vectorial angle in the complex plane, then from the Euler identity $e^{i\Theta} = \cos\Theta + i\sin\Theta$, we see that e^{iyx} represents a point on the unit circle in this plane. Since re^{iyx} represents a point on the circle whose radius is r, the geometrical effect of multiplying e^{iyx} by $f(x)$ is to move the point along the terminal side of Θ. The modulus of the vector $e^{iyx}f(x)$ is less than unity because

$$\sum_x f(x) = 1.$$

When all such vectors, corresponding to the various values of x, are added according to the so-called parallelogram law by matching the head of one vector to the tail of the next, the resulting end point will lie inside or on the unit circle.

For the continuous case,

$$M(iy,x) = \int_{-\infty}^{\infty} e^{iyx} f(x)dx. \tag{6}$$

From (2) and the fact that $f(x) \geqq 0$, the integral in (6) is convergent for any real y, and $|M(iy,x)| \leqq 1$.

Examples

1. Find the mean and variance of the distribution

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \quad -\infty \leqq x \leqq \infty.$$

Solution. It is readily verified that

$$\int f(x)dx = 1, \quad \nu_1 = \int xf(x)dx = 0,$$

and

$$\sigma^2 = \int x^2 f(x)dx = 1/(2h^2);$$

whence $h^2=1/(2\sigma^2)$. A variate whose distribution function is

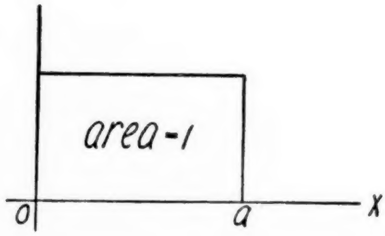
$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-x^2/2\sigma^2},\qquad -\infty\geqq x\geqq\infty,$$

is said to be *normally distributed with mean* $=0$ *and variance* $=\sigma^2$.

2. Determine the characteristic function of the rectangular (or uniform) distribution $f(x)=1/a,\quad 0\leqq x\leqq a$.

Solution. We have

$$M(w,x)=\frac{1}{a}\int_0^a e^{wx}dx$$
$$=(e^{aw}-1)/aw.$$



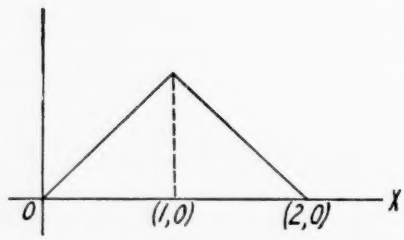
3. Determine the characteristic function for the triangular distribution defined by the line $y=x$ on the interval $0\leqq x\leqq 1$ and by the line $y=2-x$ on the interval $1\leqq x\leqq 2$.

Solution. We observe first that the distribution function satisfies (2) since

$$\int_0^1 xdx+\int_1^2(2-x)dx=1.$$

Its characteristic function is

$$M(w,x)=\int_0^1 e^{wx}xdx+\int_1^2(2-x)e^{wx}dx$$
$$=(e^{2w}-2e^w+1)/w^2.$$



Incidentally, this is the square of the result for Example 2 when $a=1$.

4. Find the characteristic function of the distribution, sometimes called the double exponential function, $f(x)=\frac{1}{2}e^{-|x|}$, $-\infty \leqq x \leqq \infty$.

Solution.

$$M(w,x)=\int_{-\infty}^{\infty} e^{wx}f(x)dx$$

$$=\tfrac{1}{2}\int_{-\infty}^{0} e^{x(1+w)}dx+\tfrac{1}{2}\int_{0}^{\infty} e^{-x(1-w)}dx,$$

since $-|x|$ equals x on the negative part of the interval and equals $-x$ on the positive part. Evaluating the above integrals we obtain

$$M(w,x)=\tfrac{1}{2}\left(\frac{1}{1+w}+\frac{1}{1-w}\right)=\frac{1}{1-w^2}.$$

If $w=iy$,

$$M(iy,x)=\frac{1}{1+y^2}.$$

5. Determine the characteristic function of the normal distribution.

Solution. From (3) and Example 1,

$$M(w,x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{wx-x^2/2\sigma^2}dx.$$

To evaluate this integral we write $x^2-2\sigma^2wx=(x-w\sigma^2)^2-(w\sigma^2)^2$.

Then

$$M(w,x)=e^{w^2\sigma^2/2},$$

since

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-P}dx=1, \quad \text{where } P=(x-w\sigma^2)^2/2\sigma^2.$$

4. *Properties.* The following statement can be proved rigorously[7] but not at the level of this exposition.

A distribution is uniquely determined by its characterisitic function. We shall call this the *uniqueness axiom.* We shall also adopt as an axiom a famous duality which is known as a Fourier transform,* namely,

$$
(7) \qquad \begin{cases} M(iy) = \displaystyle\int_{-\infty}^{\infty} e^{iyx} f(x)dx \\[2ex] f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} M(iy)e^{-iyx}dy, \end{cases}
$$

where $M(iy)$ is shorter notation for $M(iy,x)$.

Example 6. Determine the distribution whose characterisitic function is

$$M(iy) = e^{-y^2\sigma^2}/2.$$

Solution. Utilizing (7) we obtain

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(y^2\sigma^2+2iyx)/2}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-[(y\sigma+ix)^2/2\sigma^2]} \times e^{-(x^2/2\sigma^2)}dy \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(u^2/2\sigma^2)}du \times \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x^2/2\sigma^2)} \\
&= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x^2/2\sigma^2)}.
\end{aligned}
$$

Naturally, we find a normal distribution with variance σ^2 (See Example 5).

The uniqueness and duality axioms can be used to evaluate many definite integrals. As an example, consider the problem of evaluating

$$\int_{-\infty}^{\infty} \frac{e^{iyx}dx}{\pi(1+x^2)}.$$

*An expository article on Fourier transforms has recently been published by Cameron.[8]

We recognize this integral as the result of setting up the characteristic function of the Cauchy distribution. Also we recall (from Example 4) that the characteristic function of the double exponential function is $1/(1+y^2)$. We conjecture, therefore, that the answer to our problem is

$$M(iy)=e^{-|iy|}, \quad |iy|=|y|.$$

Let us test our guess by means of the Fourier transform. Substitution of the above expression into the appropriate part of (7) yields

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|y|}e^{-iyx}dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{y(1-ix)}dy + \frac{1}{2\pi}\int_{0}^{\infty} e^{-y(1-ix)}dy \\ &= \frac{1}{2\pi}\left(\frac{1}{1-ix}+\frac{1}{1+ix}\right) \\ &= \frac{1}{\pi(1+x^2)}. \end{aligned}$$

Thus our guess is verified and we have

$$\int_{-\infty}^{\infty}\frac{e^{iyx}dx}{\pi(1+x^2)}=e^{-|y|}$$

an integral due to Laplace.

Upon expanding e^{wx} into a power series we have, from (4),

$$M(w,x)=E\left(1+wx+\frac{w^2}{2!}x^2+\frac{w^3}{3!}x^3+\cdots\right)$$

whence, with the aid of Proposition 1, we obtain

$$M(w,x)=1+w\nu_1+\frac{w^2}{2!}\nu_2+\frac{w^3}{3!}\nu_3+\cdots, \tag{8}$$

provided the moments exist. In case the origin of x is at ν_1 (as in Example 5), the ν's in (8) are to be replaced by μ's of corresponding order. Thus, $M(w,x)$ may be regarded as *generating* the moments in the sense that ν_r (or μ_r) is the coefficient of $w^r/r!$ in the expansion above. The variable w is sometimes said to *carry* the moments.

If the rth derivative of $M(w,x)$ with respect to w exists at $w=0$ we obtain from (8),

$$M^{(r)}(0,x)=\nu_r, \tag{9}$$

where here and subsequently the left member denotes the rth derivative of $M(w,x)$ at $w=0$. If the origin of x is chosen so that $E(x)=0$, then the right member of (9) becomes μ_r. Therefore, the characteristic function has the important property that its rth derivative at the origin of w gives the rth moment of the distribution of x. If w is replaced by iy, the factor i^r will appear in the right member of (9).

Example 7. Find the moments ν_r of the rectangular distribution $f(x)=1/a$.

Solution. Although the moments of this distribution can be obtained easily from the definition in §2 we shall use this example to illustrate (8). Referring to Example 2, we have

$$\frac{e^{wa}-1}{wa}=1+\frac{wa}{2!}+\frac{w^2a^2}{3!}+\frac{w^3a^3}{4!}+\cdots$$

and comparing this with (8) we see that

$$\nu_r=\frac{a^r}{r+1}.$$

Example 8. Find the variance of the discrete distribution $f(x)=1/n$, $x=1,2,\cdots,n$. (Unbiased dice illustrate this distribution for the case $n=6$.)

Solution. We note first that $\sum_x f(x)=1$. Then from (4),

$$\begin{aligned}M(w,x)&=\frac{1}{n}\sum_x e^{wx}\\&=\frac{1}{n}(e^w+e^{2w}+\cdots+e^{nw})\\&=1+S_1w+S_2\frac{w^2}{2!}+S_3\frac{w^3}{3!}+\cdots,\end{aligned}$$

where $S_r=\frac{1}{n}\sum_x x^r$. Comparison with (8) shows that

$$\nu_1=S_1=(n+1)/2,\quad \nu_2=S_2=(n+1)(2n+1)/6.$$

Using the relation $\mu_2=\nu_2-\nu_1^2$ we find $\sigma^2=(n^2-1)/12$.

Example 9. Determine the moments μ_r of the normal distribution.

Solution. From Example 5, we have

$$M(w,x)=e^{w^2\sigma^2/2}=1+\left(\frac{\sigma^2}{2}\right)w^2+\frac{1}{2!}\left(\frac{\sigma^2}{2}\right)^2w^4+\frac{1}{3!}\left(\frac{\sigma^2}{2}\right)^3w^6+\cdots$$

whence we obtain

$$\mu_{2r+1}=0, \quad \mu_{2r}=1\cdot 3\cdot 5\cdots(2r-1)\sigma^{2r}.$$

In particular, we note that $\mu_3=0$ and $\mu_4=3\sigma^4=3\mu_2{}^2$.

Two simple but useful relations are the following:

$$M(w,cx)=M(cw,x) \tag{10}$$

$$M(w,x-c)=e^{-cw}M(w,x), \tag{11}$$

where c is an arbitrary constant. These properties follow from (4). The first states that a change of scale, so that x becomes cx, has the effect of replacing w by cw in the characteristic function. The second says that a change of origin from $x=0$ to $x=c$ has the effect of multiplying the characteristic function by the factor e^{-cw}.

Repeated differentiation of (10) yields, in accord with (9),

$$\nu_{r:cx}=c^r\nu_{r:x} \tag{12}$$ or

$$\mu_{r:cx}=c^r\mu_{r:x}\,, \tag{12a}$$

according as $E(x)=\nu_1$ or $E(x)=0$.

If $x_1, x_2, \cdots, x_N$ are N mutually independent variates, and if $L=c_1x_1+c_2x_2+\cdots+c_Nx_N$, where here and hereafter the c_j's are arbitrary constants not all zero, then Proposition 2 enables us to write the important formula

$$\begin{aligned} M(w,L)=E(e^{wL}) &= E\left\{\prod_{j=1}^{N} e^{wc_jx_j}\right\} \\ &= \prod_{j=1}^{N} E(e^{wc_jx_j}) = \prod_{j=1}^{N} M(w,c_jx_j), \end{aligned} \tag{13}$$

where $M(w,c_jx_j)$ is the characteristic function of c_jx_j. From (10) and (13) we have

$$M(w,L)=\prod_{j=1}^{N} M(c_jw,\, x_j). \tag{14}$$

5. *The binomial distribution.* A variate x is said to have a binomial distribution with $n+1$ values and parameter p if

$$f(x)=C(n,x)p^xq^{n-x}, \quad x=0,1,2,\cdots,n, \tag{15}$$

where $q=1-p$. It is often called the Bernoulli distribution, after Jacob Bernoulli who first studied it in detail. His research was written in Latin and published, posthumously, by his brothers in 1713 under the title *Ars Conjectandi*. The characteristic function of (15) is

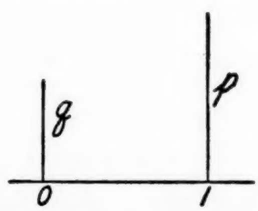
$$M(w,x)=\sum_{x} C(n,x)(pe^{w})^{x}q^{n-x} \tag{16}$$
$$=(q+pe^{w})^{n}.$$

The moments ν_r can be obtained in terms of n and p by differentiating the above expression and setting $w=0$ in the various derivatives. For example, from (9), we obtain for $r=1$ and $r=2$,

$$\nu_1=np, \quad \nu_2=n(n-1)p^2+np.$$

For certain purposes it is convenient to express x as follows:

$$x=z_1+z_2+\cdots+z_n, \tag{17}$$



where the z's are identical variates, each with the two values 0 and 1 assumed with respective probabilities q and p. If x is interpreted as the number of white balls obtained in a set of n drawings from an urn containing white and black balls, when p is the probability of drawing a white ball in each drawing, then z_j would represent the result of the jth drawing. Obviously, $E(z_j)=p$. The characteristic function of any one of the z_j's is, from (4),

$$M(w,z_j)=e^{0}q+e^{w}p=(q+pe^{w}), \tag{18}$$

whence, from (13) with x in the role of L and z_j in place of x_j, we obtain

$$M(w,x)=\prod_{j=1}^{n} M(w,z_j)=(q+pe^{w})^{n}$$

which agrees with (16) as, of course, it should.

A neat derivation of the well known formula $\sigma_x^2=npq$ can be given by use of (17). The variance of z_j is, by definition,

$$\sigma_j^2=(0-p)^2q+(1-p)^2p$$
$$=p^2q+q^2p=pq.$$

Then by the additive property of variances (of independent variates) we have

$$\sigma_x^2 = \sum_{j=1}^{n} \sigma_j^2 = npq.$$

The limiting form of the binomial distribution, as $n \to \infty$, is the normal curve. This will be stated more precisely and proved in the DeMoivre-Laplace theorem (§13). As a consequence of this theorem, the sum of the terms of $(q+p)^n$ for which $a < x < b$ can be approximated by finding the appropriate area under a normal curve. (The procedure by which this is accomplished is usually explained in textbooks[9] on mathematical statistics.) If p (or q) is extremely small, however, the approximation is usually not good. If p (or q) is of the order of magnitude of $1/n$, denoted by $(0)1/n$, then a more satisfactory approximation to the terms of the binomial is given by the Poisson exponential function (§7).

6. *Generating functions and factorial moment generating functions.* In the right member of (4) let* $w = \log v$ and denote by $G(v,x)$ the result of the substitution there. Then

$$G(v,x) = \sum_x v^x f(x) \qquad \text{or} \qquad \int_{-\infty}^{\infty} v^x f(x)\,dx. \tag{19}$$

We shall call $G(v,x)$ the *generating function*† (g. f.) associated with the probability function $f(x)$.

If now we replace v by $1+u$ in the discrete case above, we obtain the *factorial moment generating function* (f. m. g. f.) denoted by $H(u,x)$. Thus

$$H(u,x) = \sum_x (1+u)^x f(x). \tag{20}$$

Upon expanding $(1+u)^x$ and summing, we have

$$H(u,x) = 1 + \nu_{(1)}u + \nu_{(2)}\frac{u^2}{2!} + \nu_{(3)}\frac{u^3}{3!} + \cdots,$$

where $\nu_{(r)} = \sum_x x^{(r)} f(x), \quad x^{(r)} = x(x-1)(x-2)\cdots(x-r+1),$

and $\nu_{(r)}$ is called the *factorial moment* (of order r) of the distribution $f(x)$.

*All logarithms in this paper are to the base e.

†An interesting booklet by Soper[10] on generating functions was published in 1922. See also Aitken's book.[6]

7. *The Poisson distribution.* We return to the binomial to consider the case in which p is small and n is large. If the mean np remains finite as $p \to 0$ and $n \to \infty$, the limit of the binomial is the Poisson distribution. To exhibit this limit we proceed as follows.

The generating function of $f(x)$ is obtained from its characteristic function by replacing e^{w} by v. Thus from (16) we have

$$G(v,x) = (q+pv)^n,$$

whence the f. m. g. f. of the binomial distribution is seen to be

$$H(u,x) = 1+pu)^n.$$

Let $m = np$. Then $\quad H(u,x) = (1+mu/n)^n$

and $\quad \lim_{n \to \infty} (1+mu/n)^n = e^{mu}.$

This result is the f. m. g. f. of a probability function $f(x)$. Its g. f. is obtained by replacing u by $v-1$, so we have

$$G(v,x) = e^{m(v-1)}.$$

It is evident from (19) that the coefficient of v^x in the g. f. is $f(x)$. The coefficient of v^x in the expansion of $e^{m(v-1)}$ is

$$f(x) = e^{-m}m^x/x!\,.$$

This is the Poisson distribution of rare statistical events. Many applications of this function in telephone traffic and other fields are given in an interesting book by Fry.[11]

Example 10. (a) If x has a Poisson distribution, evaluate

$$\sum_{x=0}^{\infty} f(x).$$

(b) Taking $m=2$, make a table of (approximate) values of $f(x)$ for $x=0, 1, 3, 4, 5$.

Solution.

$$e^{-m} \sum_{x=0}^{\infty} m^x/x! = 1$$

x	0	1	8	3	4	5	*Sum*
$f(x)$	.14	.28	.28	.18	.09	.02	.99

8. *Cumulant characteristic functions.* Relation (13) suggests that for certain purposes the logarithm of $M(w,x)$ might be more useful than M itself. Thus, we define

$$K(w,x) = \log M(w,x). \tag{21}$$

If this can be expanded into a power series of the form

$$K(w,x) = \kappa_1 w + \kappa_2 w^2/2! + \kappa_3 w^3/3! + \kappa_4 w^4/4! + \cdots \tag{22}$$

which converges for some range of w containing the origin as an interior point, then $K(w,x)$ is called the ***cumulant characteristic function*** of the distribution of x. The κ's (kappas) are functions of the moments. To determine them we proceed as follows.

Since $K^{(1)} = M^{(1)}/M$ we have $M \cdot K^{(1)} = M^{(1)}$. From the latter relation, together with (8) and (22) we get

$$\left(1 + \nu_1 w + \nu_2 \frac{w^2}{2!} + \nu^3 \frac{w^3}{3!} + \cdots\right)\left(\kappa_1 + \kappa_2 w + \kappa^3 \frac{w^2}{2!} + \kappa^4 \frac{w^3}{3!} + \cdots\right)$$

$$= \nu_1 + \nu_2 w + \nu^3 \frac{w^2}{2!} + \nu^4 \frac{w^3}{3!} + \cdots,$$

and upon equating coefficients of like powers of w, we have

$$\begin{cases} \kappa_1 & = \nu_1 \\ \nu_1 \kappa_1 + \kappa_2 & = \nu_2 \\ \nu_2 \kappa_1 + 2\nu_1 \kappa_2 + \kappa_3 & = \nu_3 \\ \nu_3 \kappa_1 + 3\nu_2 \kappa_2 + 3\nu_1 \kappa_3 + \kappa_4 & = \nu_4 \\ \dots\dots\dots\dots\dots\dots & \end{cases} \tag{23}$$

In (23) there is an infinite sequence of equations, each of which contains one more term than the preceding; in each of the left members the binomial coefficients prevail and the subscripts on the ν's decrease while those on the κ's increase; the right member of the rth equation is ν_r.

By inspection, $\kappa_1 = \nu_1, \quad \kappa_2 = \nu_2 - \nu_1^2 = \mu_2.$

Utilizing these results in solving the first three equations of (23), we obtain $\kappa_3 = A/B$, where

$$B = \begin{vmatrix} 1 & 0 & 0 \\ \nu_1 & 1 & 0 \\ \nu_2 & 2\nu_1 & 1 \end{vmatrix} = 1, \qquad A = \begin{vmatrix} 1 & 0 & \nu_1 \\ \nu_1 & 1 & \nu_2 \\ \nu_2 & 2\nu_1 & \nu_3 \end{vmatrix}$$

Hence, $\kappa_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 = \mu_3.$

At this point one might wonder if $\kappa_r = \mu_r$ for $r > 3$. The answer is negative. Determination of the kappas of higher order is facilitated by simplifying the equations in (23). This can be accomplished, without loss of generality, by a translation of the origin of x. Let us apply (11) taking $c = \nu_1$ and letting $x' = x - \nu_1$. Then

$$
\begin{aligned}
(24) \qquad K(w,x') &= -\nu_1 w + K(w,x) \\
&= \kappa_2 w^2/2! + \kappa_3 w^3/3! + \kappa_4 w^4/4! + \cdots .
\end{aligned}
$$

The effect of the translation is to make $\kappa_1 = 0$ in (24) and to replace the ν's by μ's in (23). So the first four equations of (23) become

$$
\begin{aligned}
&\kappa_1 &&= 0 \\
&0 + \kappa_2 &&= \mu_2 \\
&\mu_2 \kappa_1 + 0 + \kappa_3 &&= \mu^3 \\
&\mu_3 \kappa_1 + 3\mu_2 \kappa_2 + 0 + \kappa_4 &&= \mu_4.
\end{aligned}
$$

Solving for κ_3 and κ_4 we find,

$$
\kappa_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu_2 \\ \mu_2 & 0 & \mu_3 \end{vmatrix} = \mu^3
$$

$$
\kappa_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu_2 \\ \mu_2 & 0 & 1 & \mu^3 \\ \mu_3 & 3\mu_2 & 0 & \mu_4 \end{vmatrix} = \mu^3 - 3\mu_2^2.
$$

The translation changes no kappa except the first. The values of κ_r for $r > 4$ in terms of the μ's can be obtained in a similar manner but we shall not need them. Expressing the first four moments in terms of the kappas, we have

$$
(25) \qquad \begin{cases} \nu_1 = \kappa_1, & \mu_3 = \kappa_3 \\ \mu_2 = \kappa_2, & \mu_4 = \kappa_4 + 3\kappa_2^2. \end{cases}
$$

The name *cumulants* for the kappas seems to have been suggested by Hotelling[12] and has been adopted by Fisher[13] and his school.*

From (22) it follows that

$$
(26) \qquad K^{(r)}(0,x) = \kappa_r.
$$

*Other writers call the κ's "semi-invariants" in accord with the nomenclature used by Thiele[14] who was the first to realize their importance.

Utilizing (10) and (21) in (22) we have

$$K(w,cx)=K(cw,x)$$
$$=\kappa_1(cw)+\kappa_2(cw)^2/2!+\kappa_3(cw)^3/3!+\cdots, \tag{27}$$

and from (26) and (27) we obtain

$$\kappa_{r:cx}=c^r\kappa_{r:x}\,. \tag{28}$$

This relation expresses the rth cumulant of cx in terms of the rth cumulant of x.

When independent variates are compounded by the relation

$$L=\sum_{j=1}^{N} c_jx_j,$$

formula (13) shows that the characteristic functions are compounded in product. Hence we would expect that the cumulants would be compounded in sum. Application of (21) to (13) yields

$$K(w,L)\sum_{j=1}^{N} K(w,c_jx_j) \tag{29}$$

and from this result, together with (26) and (28), we obtain the following relation between the cumulants of L and those of the x_j's:

$$\kappa_{r:L}=\sum_{j=1}^{N} c_j{}^r\kappa_{r:x}\,. \tag{30}$$

This additive property of cumulants is indeed their chief *raison d'etre.*

From (14) and (29),

$$K(w,L)=\sum_{j=1}^{N} K(c_jw,x_j)\,. \tag{29a}$$

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Writers on technical or educational mathematics within the range of programs designed to prepare applicants for naval or air service with the armed forces of the United States are invited to submit their papers for publication in this journal.

Summation of Finite Series with Polynomial Terms

By Albert B. Farnell
University of California

To simplify our notation in the following, we introduce

$$\Delta h(n) = h(n+1) - h(n).$$

Now suppose we have given a series whose nth term is of the form $g(n)$, a polynomial of degree $m-1$ in n. Let $f(n)$ denote the sum of the first n terms of this series. We have then,

$$f(n) + g(n+1) = f(n+1),$$

or

$$g(n+1) = f(n+1) - f(n)$$

$$= \Delta f(n).$$

As will be seen, it is sufficient to assume $f(n)$ a polynomial of degree m in n, for the coefficients of any higher powers of n would be zero. Let then

$$f(n) = a_0 n^m + a_1 n^{m-1} + \cdots + a_{m-1} n + a_m.$$

To satisfy our induction hypothesis, we have only to set

$$f(1) = g(1) = a_0 + a_1 + \cdots + a_{m-1} + a_m.$$

Further, we have

$$g(n+1) = a_0 \Delta n^m + a_1 \Delta n^{m-1} + \cdots + a_{m-1} \Delta n. \tag{1}$$

On the right we have m unknowns, which are readily obtained on equating the coefficients of like powers of n.

The coefficients of various powers of n on the right can be directly obtained from the Pascal triangle. Moreover, as is easily seen, $a_0 + a_1 + \cdots + a_{m-1}$ is equated to the constant term of $g(n+1)$, so that $a_m = 0$ always. Hence

$$f(n) = a_0 n^m + a_1 n^{m-1} \cdots + a_{m-1} n,$$

where the coefficients a_i are determined from (1).

Let us determine the sum of a series whose nth term is of the form $n^3 + 3n^2 + 4$.

We set up the Pascal triangle and proceed as indicated:

$$
\begin{array}{cc|ccccc}
 & 1 & & & & & \\
a_3 & 1 & 1 & & & & \\
a_2 & 1 & 2 & 1 & & & \\
a_1 & 1 & 3 & 3 & 1 & & \\
a_0 & 1 & 4 & 6 & 4 & 1 & \\
\hline
 & 1 & 5 & 10 & 10 & 5 & 1 \\
 & \cdot\cdot & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

From this we can write down directly

$$
\begin{aligned}
(4a_0)n^3+(6a_0+3a_1)n^2+(4a_0+3a_1+2a_2)n+(a_0+a_1+a_2+a_3) &= (n+1)^3+3(n+1)^2+4 \\
&= n^3+6n^2+9n+8.
\end{aligned}
$$

Equating coefficients we have

$$
\begin{aligned}
4a_0 &= 1 \\
6a_0+3a_1 &= 6 \\
4a_0+3a_1+2a_2 &= 9 \\
a_0+a_1+a_2+a_3 &= 8.
\end{aligned}
$$

From these, we find $a_0={}^1/_4$, $a_1={}^3/_2$, $a_2={}^7/_4$, $a_3={}^9/_2$. Therefore

$$
\begin{aligned}
f(n) &= {}^1/_4\, n^4+{}^3/_2\, n^3+{}^7/_4\, n^2+{}^9/_2\, n \\
&= {}^1/_4\, n(n^3+6n^2+7n+18).
\end{aligned}
$$

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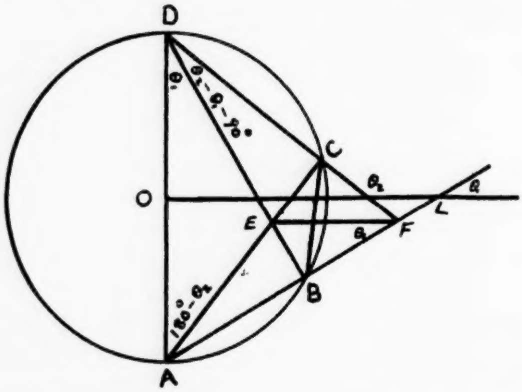
Note on a Quadrilateral Inscribed in a Semicircle

By F. A. Lewis
University of Alabama

Let the quadrilateral $ABCD$ be inscribed in a semicircle of diameter $AD=a$. Denote the angles which AB and CD make with the



diameter perpendicular to AD by θ_1 and θ_2 respectively. One may easily verify that the various angles have the values indicated in the figure. Therefore, we have

$$AD=a, \quad AB=a\cdot\sin\theta_1, \quad BD=a\cdot\cos\theta_1, \qquad (1)$$

$$CD=a\cdot\sin\theta_2, \quad AC=-a\cdot\cos\theta_2.$$

All the line lengths in the figure, including some that are not lettered, are readily expressible in terms of a and the trigonometric functions of θ_1 and θ_2, and hence in terms of the five parts of the quadrilateral $ABCD$ appearing in (1). For example, in the triangle BCD we find $BC=-a\cdot\cos(\theta_1-\theta_2)$, from which Ptolemy's theorem for this special case follows. This is only one of more than a dozen similar relations which are easily proved. A few typical illustrations follow.

$$BF=BD\tan(\theta_2-\theta_1-90°)=a\cos\theta_1\cdot\cot(\theta_1-\theta_2),$$

so that EF (which is seen to be perpendicular to AD) has the value $a\cdot\cot(\theta_1-\theta_2)$. Expressed in terms of line lengths this gives

$$\frac{EF}{AD}=\frac{BD\cdot AC-AB\cdot CD}{AB\cdot AC+BD\cdot CD}. \qquad (2)$$

From the right triangle AOL, we get

$$2\,OL \cdot AB = AD \cdot BD. \tag{3}$$

From $$DE = \frac{AD \sin \Theta_2}{\sin(\Theta_2 - \Theta_1)},$$ we get

$$\overline{AD}^2 \cdot CD = DE(AB \cdot AC + BD \cdot CD). \tag{4}$$

Conversely, known relations which exist between the various lines in the figure furnish interesting trigonometric identities. The relation $AD^3 - (AB^2 + \overline{BC}^2 + \overline{CD}^2)AD - 2 \cdot AB \cdot BC \cdot CD = 0$* gives

$$\sin^2\Theta_1 + \sin^2\Theta_2 + \cos^2(\Theta_1 - \Theta_2) - 2 \cdot \sin\Theta_1 \cdot \sin\Theta_2 \cdot \cos(\Theta_1 - \Theta_2) = 1. \tag{5}$$

The theorem on the ratio of the diagonals of a cyclic quadrilateral when applied to $ABCD$ and $BFCE$ respectively gives

$$-\frac{\cos\Theta_1}{\cos\Theta_2} = \frac{\sin\Theta_2 - \sin\Theta_1 \cdot \cos(\Theta_1 - \Theta_2)}{\sin\Theta_1 - \sin\Theta_2 \cdot \cos(\Theta_1 - \Theta_2)} \tag{6}$$

$$\sin(\Theta_1 - \Theta_2) = \frac{\sin\Theta_2 \cos\Theta_2 - \sin\Theta_1 \cos\Theta_1}{\sin\Theta_1 \sin\Theta_2 - \cos\Theta_1 \cos\Theta_2}. \tag{7}$$

If we only assume that $ABCD$ is inscribed in a circle and label the angles $BDA = \alpha$, $CAD = \beta$, $CAB = \gamma$, $ABD = \delta$, the law of sines combined first with Ptolemy's theorem and then with the theorem on the ratio of the diagonals of a cyclic quadrilateral furnishes the following relations which hold for all angles $\alpha + \beta + \gamma + \delta = 180°$.

$$\sin\alpha \cdot \sin\beta + \sin\gamma \cdot \sin\delta = \sin(\alpha + \gamma) \cdot \sin(\beta + \gamma), \tag{8}$$

$$\frac{\sin(\beta + \delta)}{\sin(\beta + \gamma)} = \frac{\sin\alpha \cdot \sin\delta + \sin\beta \cdot \sin\gamma}{\sin\alpha \cdot \sin\gamma + \sin\beta \cdot \sin\delta}. \tag{9}$$

*See Dickson, *First Course in the Theory of Equations*, page 100, ex. 16. *New First Course*, page 95, ex. 42.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

LETTER TO THE EDITOR

UNIVERSITY OF ILLINOIS,
November 2, 1942.

My dear Professor Sanders:

The October number of the NATIONAL MATHEMATICS MAGAZINE (page 15) contains a reference to the recent *Survey of Modern Algebra,* by Birkhoff and MacLane (1941) which may be too brief to be clear to many readers. It should be noted that it relates to a question in the history of mathematics and hence it is not fundamentally connected with the main subject of the volume in question. In a review of this volume which appeared in the April (1942) number of the widely consulted *Mathematical Reviews,* page 100, it is asserted that this volume "seems to be free from error." This is a somewhat unusual claim for a mathematical book of considerable size and makes it more desirable to consider evidences which seem to point to its inaccuracy, especially since this book will probably be widely consulted in view of its many good points.

The main statements to which the given reference relate are the following: "A completely geometric approach to real numbers was used by the Greeks. For them a number was simply a ratio $(a:b)$ between two line segments a and b." On the contrary, in Book X of Euclid's *Elements,* it is asserted that incommensurable quantities do not have the same ratio to each other as two numbers. This is one of the clearest statements of the sharp distinction which the ancient Greeks make between numbers and geometric quantities. In the seventeenth century I. Newton and others emphasized the identity between numbers and line segments as is explained on page 93 (1933) of volume 2, Tropfke's *Geschichte der Elementar-Mathematik.* As it is not now generally believed that Pythagoras discovered the irrational quantities the term "Dilemma of Pythagoras" (page 61) is also misleading.

Sincerely yours,

G. A. MILLER.

Laplace's Contributions to Pure Mathematics

By A. W. Richeson

Pierre Simon Laplace was born in Normandy, France, in 1749 and died in 1827. Very little is known of his youth, since in after life he refused to speak of his childhood days. At eighteen through the aid of D'Alambert he secured a position as professor of mathematics at the Ecole Militaire of Paris. He was elected a member of the French Academy and of the Academy of Sciences; later he was made President of the Bureau of Longitude. It was during this time that he carried on the greater part of his scientific researches that gained for him the title of "the Newton of France."*

Laplace gave to the world three great works: the *Mècanique Céleste*, the *Exposition du Système du Monde*, and the *Théorie Analytique des Probabilités*. Besides these he presented numerous important memoires before the French Academy and the Academy of Sciences. Laplace's works were first published in seven volumes by the French government in 1843.

The *Exposition du Système du Monde*, a non-mathematical popular treatment of Astronomy, was published in 1796. The *Mècanique Céleste*, published in 1799, was a complete analytical solution of the mechanical problems presented by the solar system. In 1812 he published a series of papers of which the main results were later collected and called the *Théorie Analytique des Probabilités*. This work contains the greater part of his researches in pure mathematics. The third edition, published in 1820, consisted of an introduction and two books. The introduction is an exposition of the principles and application of the science of probabilities without the aid of analytical formulas. The first book gives the theory of generating functions which are applied in the second book to the theory of probabilities.

In this work on probabilities Laplace gave a method of approximating the value of definite integrals; the solution of differential equations; the use of difference equations; and an important part of the work is devoted to the application of probabilities to the method of least squares and the theory of errors.

*Cajori, *History of Mathematics*, (New York) 1919, pp. 259-266. Ball, *A Short History of Mathematics*, (London) 1919, pp. 412-421.

In discussing his contributions to pure mathematics in detail we find the following were the most important:

In differential equations he showed that the potential function always satisfied the partial differential equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{dy^2}+\frac{\partial^2 v}{\partial z^2}=0.$$

This is known as Laplace's equation and was first given by him in this and in a more complicated form in polar coordinates. He also gave a method for the transformation of the linear differential equation of the form $Rr+Ss+Tt+Pp+Qq+Zz=U$, where R, S, T, P, Q, Z and U are functions of x and y, to a simpler form and thereby a method of solution. We then find a method for the integration of an equation of the second order which is linear in the dependent variable and its derivatives. This is followed by a theorem which states that the solution of a given differential equation with rational coefficients can be obtained in the form of a definite integral. The greater part of his work with differential equations is found in the *Mècanique Céleste* and a memoire read before the French Academy.

In the course of his work, especially in his researches on *the Exposition du Système du Monde,* Laplace had occasion to eliminate n quantities from a set of linear equations. His methods of obtaining the unknown solutions are given in a memoire published in the *History of the Royal Academy of Science* in 1772. He obtains the solution of the unknown by means of determinants and introduces the term *resultant* for the expression which gives the value of the unknown. He then gives a proof of a theorem of Vandermonde on the effect of transposing two elements in the resultant. He further shows that when two elements are alike the resultant vanishes. The mode of elimination and the solution of the unknowns in simultaneous equations was practically the same as that used today.

The most important part of the memoire is given to the method of calculation of the resultant. It is introduced as "a very simple process for considerably abridging the calculation of the equation of condition between a, b, c, &."* As a matter of fact it was of more importance than this as it involves a general expansion theorem to which Laplace's name is attached.

The calculation of the resultant necessitated a notation which Laplace supplied as follows: "I designate by $(a\ b\ c)$ the quantity $abc-acb+cab-bac+bca-cba$ and by $(a\ b)$ the quantity $ab-ba$, etc.; by $({}^1a\ {}^2b\ {}^3c)$ I indicate the quantity $(a\ b\ c)$ in the terms from which

*Muir, *History of Determinants*, Vol. I, (London), 1906, p. 27.

we give 1 to the index to the first letter, 2 for the second, and 3 for the third; by $({}^1 a\ {}^2 b)$ I designate the quantity $(a\ b)$ in the terms from which we give 1 to the index to the first letter and 2 to the second; etc."* Here again his notation is practically the same as that used today. He later solves the problem of determining the number of terms in this expansion.

The sum of Laplace's contributions to the theory of determinants may be stated as follows:

1. The name resultant for the expression for the value of the unknown.
2. A proof of the theorem regarding the effect of the transposition of two adjacent elements in the resultant.
3. A proof of the theorem regarding the effect of equalizing two of the elements of the resultant.
4. A notation for a resultant, e. g. $({}^1 a\ {}^2 b\ {}^3 c)$.
5. A method of arriving at the solution of a set of simultaneous equations.
6. A rule for expressing a resultant as an aggregate of terms composed of factors which are themselves resultants.
7. A method for finding the number of terms in this aggregate.

Laplace's notion of the generating function was introduced in his *Théorie Analytique des Probabilités.* It consists of treating the successive values of any function as the coefficients in the expansion of another function with reference to a different variable. The latter function is called the generating function of the first. He then shows that the coefficients may be determined by a method of interpolation from the generating function. About this function Laplace says: "Given y_x any function of x; if we form the infinite sequence

$$y_0+y_1t+y_2t^2+\cdots+y_xt^x+\cdots+y_\infty t^\infty\ ;$$

we are always able to conceive of a function of t, which, developed according to the powers of t, gives this sequence: this function is that one which I call the generating function of y_x.

The generating function of any variable y_x is then generally a function of t, which, developed according to the powers of t, has this variable for the coefficient of t^x; and reciprocally, the variable corresponding to a generating function is the coefficient of t^x in the development of this function following the powers of t. Again, the exponents, both positive and negative, of the power of y indicate the position

*Muir, *History of Determinants,* Vol. I, (London) 1906, p. 30.

which the variable occupies in the series which we may conceive of as prolonged indefinitely to the left."*

Laplace then considers the converse problem: Given the coefficients to determine the generating function. It is here that he makes use of finite differences. He was the first to show that a recurrent series is connected with the solution of an equation in finite differences with one independent variable, and also that a recurro-recurrent series is similarly connected with the solution of an equation in finite differences with two independent variables.† Laplace introduced the term *finite differences* as well as the subject itself. In a further treatment of the subject of finite difference equations he was the first to consider the difficult problem involved in the equations of mixed differences and to prove that the solution of an equation in finite differences of the first degree and second order could always be obtained in the form of a continued fraction.

In a memoire on the Theory of Probability published about 1778 Laplace considers the evaluation of many definite integrals by a method of approximation.‡ He first applies the method to an integral of the type $\int y\,dx$ where $y=x^p(1-x)^q$ the integral being taken between assigned limits. Later in this memoire Laplace develops another form of approximation of the definite integral which is as follows: Suppose it is required to evaluate $\int y\,dx$, let Y be a maximum value of y within the range of integration. Assume $y=Ye^{-t^2}$ and thus change $\int y\,dx$ into an integral with respect to t. Laplace later determines the value of

$$\int_0^\infty e^{-t}dt.$$

This is accomplished by taking the double integral

$$\int_0^\infty \int_0^\infty e^{-s(1+u)}dsdu$$

and equating the results which are obtained by considering the integration in different orders. He also considers $y=Ye^{-t^4}$ instead of $y=Ye^{-t^2}$.

Todhunter says that this memoire, on account of the method of approximating the value of definite integrals, deserves to be regarded as a great contribution to mathematics in general outside of its application to the theory of probabilities.

Théorie Analytique des Probabilités, Book I, Chap. I, pp. 9-10; Third Edition, 1820.
†Todhunter, *History of the Theory of Probabilities*, (London), 1865, pp. 464-465.
‡Todhunter, *History of the Theory of Probabilities*, (London), 1865, pp. 479-483.

Although Gauss and Legendre had previously given empirically the method of least squares for the combinations of numerous observations, Laplace gives a formal proof of the theory in the fourth chapter of the second book of the theory of probabilities.* This chapter is taken up exclusively with the theory of errors and the method of least squares. His method of reasoning for the law of errors is as follows: The normal law of errors is a first approximation to the frequency with which values are apt to be assumed by a variable magnitude on a great number of independent variables, each of which assumes different values in random fashion over a limited range according to a law of error, not in general the law of error, nor in general the same law for each variable.

This chapter seems to be the most difficult to follow in all of Laplace's work. However, Todhunter says, "Although his processes are obscure and repulsive, yet they contain all that is essential in the theory of errors and least squares."† M. Charlier remarked that among various deductions of the law of errors the exhaustive researches of Laplace occupy beyond doubt a leading place because of their generality and far reaching applications.

Among the minor discoveries of Laplace we may mention the following: About 1782 Laplace was the first to introduce integral equations into analysis.‡ He considered the equation

$$f(x) = \int e^{xt}\Phi(t)dt, \qquad g(x) = \int t^{x-L}\Phi(t)dt$$

where $\Phi(t)$ represents the unknown function in each case. The first integral equation of which a solution was obtained was Fourier's equation

$$f(x) = \int_{-\infty}^{\infty} \cos(xt)\Phi(t)dt.$$

Laplace also gave a proof to the effect that every equation of even degree must have at least one real quadratic factor. He makes use of his method of approximation to determine the coefficient of any

*Todhunter, *History of the Theory of Probabilities*, (London), 1865, pp. 560-575.
†Todhunter, *ibid.*, p. 478.
‡Whittaker and Watson, *Modern Analysis*, (Cambridge) 1920, p. 211.

term in the expansion of a high power of a certain polynomial; e. g., suppose we require the coefficient of a^l in the expansion of

$$\left\{\frac{L}{a^n}+\frac{L}{a^{n-1}}+\frac{L}{a^{n-2}}+\cdots+\frac{L}{a}+L+a+a^2+\cdots+a^{n-1}+a^n\right\}^s$$

Throughout his researches Laplace seems to have regarded analysis merely as a tool for attacking physical problems, although his ability to handle analysis was phenomenal. He took but little time to explain how certain results were obtained as long as the results were true. Dr. Bowditch who brought out an English translation of the *Mècanique Céleste* was accustomed to remark, "whenever I meet in Laplace with the words 'Thus it plainly appears', I am sure that hours and perhaps days of hard study will alone enable me to discover how it plainly appears."*

†Todhunter, *ibid.*, p. 478.

The Teachers' Department

Edited by
JOSEPH SEIDLIN, JAMES MCGIFFERT, J. S. GEORGES
and L. J. ADAMS

Teaching the Conic

By C. D. SMITH

Text books usually approach the subject of the conics from one of two points of view. Some begin with the locus of a point whose distance from a given fixed point is proportional to its distance from a fixed line. The equation which follows has a line of symmetry usually made parallel to one axis of coordinates by means of a transformation. The result is in turn simplified by another transformation. One transformation with the ratio equal to 1 gives the first standard parabola. A different transformation with the constant ratio less than 1 or greater than 1, gives the first standard ellipse in the one case and the first standard hyperbola in the other. The beginner finds the whole procedure rather involved. Consequently teachers began trying for a more direct approach to standard forms. Some of the more recent texts use a different defining property for each conic so that the results are apparently not related. In one case the distance to a fixed point equals the distance to a fixed line. And again the sum of distances to two fixed points is constant or the difference is constant and we are asked in both cases to accept $2a$ for that constant without any apparent reason. It turns out of course that $2a$ is the axis length for either curve and the bright student begins to feel that he has been slightly duped. He next finds that each curve has a fixed line corresponding to each given point and he now feels duped some more. Finally the distance to the given point and the distance to the corresponding line have a constant ratio in each case and the good student is asking why they were not all defined as one set in the first place.

The following development seems to combine the advantages of both methods and is shorter than either method. My students like it and learn it this way. A conic is the intersection of a plane surface with a conic surface. Three well defined types are commonly called

the parabola, the ellipse, and the hyperbola. Definition: The first standard conic is the locus of a point $P(x,y)$ such that the distance from point $F(p,0)$ to the distance from a line $x=-d$ is a constant $e>0$. From the definition we have

$$\sqrt{(x-p)^2+y^2}=e(x+d).$$

By squaring both sides the equation reduces to

$$x^2(1-e^2)+2x(e^2d+p)+y^2=e^2d^2-p^2.$$

Now let $e=1$ and $d=p$, and we have the parabola

$$y^2=4px. \tag{1}$$

Next let $e<1$, $e^2d+p=0$, and $e^2d^2-p^2=b^2$, and we have the ellipse

$$x^2/a^2+y^2/b^2=1, \quad \text{where } b^2=a^2(1-e^2). \tag{2}$$

Finally let $e>1$, and $e^2d^2-p^2=-b^2$, and we have the hyperbola

$$x^2/a^2-y^2/b^2=1, \quad \text{where } b^2=a^2(e^2-1). \tag{3}$$

Immediately the focus and directrix are, for the parabola, $F(p,0)$ and $x=-p$; for the ellipse, $F(\pm ae,0)$ and $x=\pm a/e$; and for the hyperbola, the same as for the ellipse. Essential properties follow as exercises, e. g. other standard forms, tangents at symmetric points meet on the axis, for the ellipse the sum of focal distances equals $2a$, for the hyperbola the difference between focal distances equals $2a$, and all the rest. Hence we have one defining equation which specializes into three standard types of conic with standard properties determined in each case by the value of e.

A Note on an Approximation to the Square of the Circle

By WALTER H. LOWSTON
New York City

In this note we show how, for a given circle, a square may be constructed whose area approximates closely to the area of the circle. The accuracy of the construction may be inferred from the fact that the relative error in the side of the square (which is proportional to $\sqrt{\pi}$) is less than $.06\%$.

Of course this result may seem tame when the seven hundred and seven place accuracy of Shank's approximation is contrasted with it; none the less it is hard to imagine any actual construction in which errors from other sources would not outweigh this one. In addition our construction is extremely simple, and the proof of our claims for it constitute an exercise in elementary trigonometry which can be followed by a high school student.

The construction may be described as follows:

In the diagram we have started with a diameter AB of the circle and drawn the radius OC perpendicular to it. The radius OD is drawn to make an angle of 45° with AB. On the line CB the point E is determined so that EB is one-fourth of CB. Then DE intersects AB in F, and the segment AF is the side of our approximating square. Its length is $(11+\sqrt{2})/7$ times the radius of the circle. This is approximately equivalent to using 1.77345 for $\sqrt{\pi}$ instead of $1.77245\cdots$.

Without loss of generality, we assume that the radius of the circle is unity. We compute the length of FB, using the law of sines in ΔEFB. Let α be the angle FEB. Observe that angle $EFB = 135° - \alpha$, so that $\sin EFB = \sin (45° + \alpha)$. We have

$$FB = EB \sin \alpha / \sin EFB = EB \sin \alpha / \sin (\alpha + 45°).$$

Since $EB = \sqrt{2}/4$, after a simple calculation we obtain

$$FB = \tfrac{1}{2} \tan \alpha / (\tan \alpha + 1)$$

The value of $\tan \alpha$ may be obtained from ΔDGE. It is seen to be GD/GE, which is

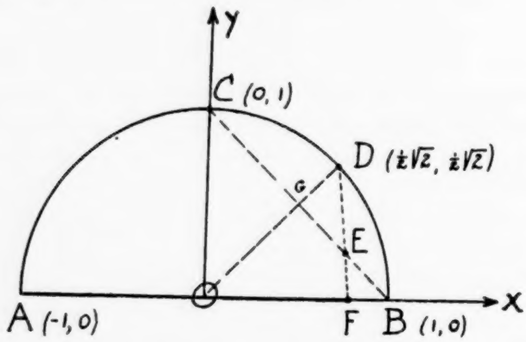
$$\left(1 - \frac{\sqrt{2}}{2}\right) \Big/ \frac{\sqrt{2}}{4} = 2\sqrt{2} - 2.$$

Thus $$FB = (3-\sqrt{2})/7.$$

Then AF, which is the line whose length we seek, is found to be

$$AF = (11+\sqrt{2}/7) = 1.77345\cdots.$$

Another method using analytics, is as follows:



$$OD = \text{bisector of } \angle BOC$$

$$EB = \tfrac{1}{4}\, CB$$

$$E : (\tfrac{3}{4}, \tfrac{1}{4})$$

Line DE : $$\frac{x-\frac{3}{4}}{\frac{1}{2}\sqrt{2}-\frac{3}{4}} = \frac{y-\frac{1}{4}}{\frac{1}{2}\sqrt{2}-\frac{1}{4}}.$$

This meets Ox at $F\left(\dfrac{4+\sqrt{2}}{7},\ 0\right)$

$$\overline{AF} = \frac{11+\sqrt{2}}{7} = \underline{1.77}345\cdots$$

$$\sqrt{\pi} = \underline{1.77}245\cdots.$$

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, West Point, N. Y.

SOLUTIONS

No. 382. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Given line AB and circles (O) and (O') on opposite sides of AB. Construct an equilateral triangle having a vertex on AB and on each of the given circles. A solution involving the method of symmetry is desired.

Discussion by *L. D. Pegg* and *J. S. B. Dick*, U. S. M. A.

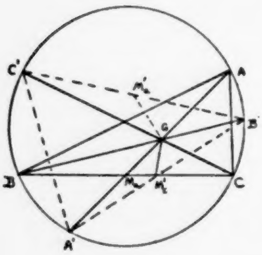
The problem has an infinitude of solutions. From some fixed point P on (O') we erect equilateral triangles PXY with X as a variable point on (O). The locus of the vertex Y is a circle (C) which is the reflection of (O) in a line making 30° with PO. The circle (C) meets AB in Z and W, two points which determine equilateral triangles with vertices on AB and (O) and at P.

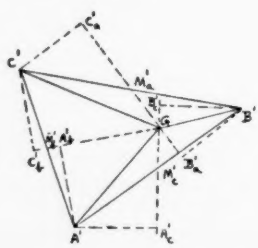
From another viewpoint, an equilateral triangle, arbitrarily fixed (within certain limits) in shape and area, moves with two of its vertices on (O) and (O'). The locus of the third vertex is the much discussed 3-bar curve. Since this curve is of the sixth degree, there are, algebraically, six possible intersections with a line AB and thus six possible positions for every selected equilateral triangle. The number of these positions depends upon the relative placement of line and circles.

No. 385. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

The medians of triangle ABC intersect its circumcircle in A', B', C'. Given these three points, construct the triangle ABC.

Solution by the *Proposer*.





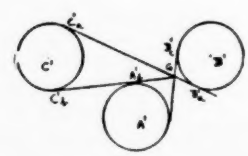


Fig. 1. If the medians AM_a, BM_b, CM_c cut the circumcircle in A', B', C' respectively and M_a', M'_b, M_c' are the midpoints of $B'C'$, $A'C'$ and $A'B'$ respectively, then triangles BCG and $B'C'G'$ are similar and GM_a and GM_a' are corresponding lines, where G is the centroid of triangle ABC.

Thus $\angle BGM_a = \angle B'GA = \angle C'GM_a'$ and GA is collinear with the symmedian of triangle $B'GC'$. In like manner, GB and GC are collinear with the symmedians from G of triangles $GC'A$ and $GA'B'$ respectively. Hence $\angle B'GM_a' = \angle AGC' = \angle A'GC = \angle B'GM_c'$ and B' lies on the bisector of $\angle M_a'GM_c'$. Likewise, C' and A' are on the bisectors of angles $M_a'GM_b'$ and $M_b'GM_c'$.

Fig. 2. Let A_b' and A_c' be the projections of A' on GM_b' and GM_c', B_a' and B_c' those of B' on GM_a' and GM_c', C_b' and C_a' those of C' on GM_b' and GM_a'.

Since any two vertices of a triangle are equidistant from the median drawn from the third vertex we have

$$A'A_c' = A'A_b' = C'C_b' = C'C_a' = B'B_a' = B'B_c'.$$

Also $$GC_a' = GC_b', \quad GB_a' = GB_c', \quad GA_b' = GA_c'.$$

Thus G is the intersection of the common interior tangents to the pairs of equal circles of radius x having A', B', C' for centers.

With proper arrangement of signs, dependent on the relative positions of the three circles we have

$$B_a'C_a' \pm C_b'A_b' \pm A_c'B_c' = 0 \quad (1).$$

Thus for Fig. 3 we have

$$(B_a'G+GA_b'+C'A_b')-C_b'A_b'-(GA_b'+B_a'G)=0$$

$$(B_a'G+GC_a')-C_b'A_b'-(A_c'G+GB_c')=0$$

$$B_a'C_a'-C_b'A_b'-A_c'B_c'=0.$$

Let the sides of triangle $A'B'C'$ be $2a'$, $2b'$, $2c'$. Then, Fig. 2, $B_a'C_a'=2M_a'B_a'=2\sqrt{a'^2-x^2}$, $C_b'A_b'=2\sqrt{b'^2-x^2}$, $A_c'B_c'=2\sqrt{c'^2-x^2}$, and equation (1) becomes

$$\sqrt{a'^2-x^2}\pm\sqrt{b'^2-x^2}\pm\sqrt{c'^2-x^2}=0 \qquad \text{or}$$

$$3x^4-2(a'^2+b'^2+c'^2)x^2+4b'^2c'^2-(b'^2+c'^2-a'^2)^2=0. \tag{2}$$

If S = area of triangle $A'B'C'$, ω its Brocard angle, then

$$\cot\omega=\frac{a'^2+b'^2+c'^2}{4S},$$

see McClelland's Geometry of the Circle, p. 62, and

$$4b'^2c'^2-(b'^2+c'^2-a'^2)^2=16S^2.$$

Hence equation (2) becomes

$$3x^4-8Sx^2\cot\omega+16S^2=0$$

$$x^2=\frac{4S}{3}(\cot\omega\pm\sqrt{\cot^2\omega-3})$$

Knowing x, we can draw the common interior tangents to circles $A'(x)$, $B'(x)$, $C'(x)$, three intersecting in G, the other three in G'. $A'G$, $B'G$ and $C'G$ cut circle $A'B'C'$ in A, B and C.

No. 444. Proposed by *H. T. R. Aude*, Colgate University.

In the scale of three, each of the fractions which have a certain integer N for denominator requires three digits in the repetend. In the scale of six, the fractions having a second integer M for denominator require six digits in the repetend. The numbers N and M when written in the denary scale use the same digits. Find M and N.

Solution by the *Proposer*.

In the scale of three the number N will have the desired property if and only if it satisfies the congruence

$$3^3\equiv 1 \pmod{N}$$

but does not satisfy $3^e \equiv 1 \pmod{N}$, $e<3$. It follows that the number N must be either twenty-six or thirteen. This points out that the number M is either sixty-two or thirty-one to correspond.

When six is the scale, the number thirty-one will have six digits in its repetend, for $e=6$ is the least integer for which the congruence

$$6^e \equiv 1 \text{ (mod thirty-one)}$$

is true. Sixty-two is not a satisfactory value of M since the statement

$$6^e \equiv 1 \text{ (mod sixty-two)}$$

cannot hold. It follows that the number M is thirty-one and N is thirteen.

No. 453. Proposed by *W. V. Parker*, Louisiana State University.

Find all the points on the hyperbola

$$3x^2-y^2+2x-y=0$$

for which both coordinates are integers.

Note by the Editors.

The standard solution for the Pell equation

$$(6x+2)^2-3(2y+1)^2=1, \tag{1}$$

which is equivalent to the given equation, is

$$(6x+2)=[(2+\sqrt{3})^n+(2-\sqrt{3})^n]/2,$$

$$(2y+1)=\pm[(2+\sqrt{3})^n-(2-\sqrt{3})^n]/2\sqrt{3}.$$

x and y will be integers if and only if n is odd and the positive sign is taken for the right member of the first equation.

It is not difficult to set up the equations

$$x'=7x+4(y+1), \quad y'=12x+7(y+1),$$

or, solved for x and y,

$$x=7x'-4y', \quad y+1=7y'-12x',$$

such that, if (x,y) is a solution of (1), so is (x',y'), and conversely. An obvious solution is (0,0) to start with. Furthermore these recurrence formulas give all the solutions. Discussions of these matters are available in many texts on the theory of numbers, e. g. *Elementary Number Theory*, Uspensky and Heaslet, pp. 332-359; *Theory of Numbers*, Wright, pp. 36-41; *Higher Algebra*, Hall and Knight, pp. 304-310.

No. 459. Proposed by *V. Thébault*, Tennie, Sarthe, France.

In every system of enumeration with base $B>2$, where B^2+1 is prime or is the product of prime numbers to the first power, there are no other eight-digit squares of the form *ababcdcd* than those whose root is a four-digit number *mnmn*, and there are always at least two of these. What is the statement for the other values of B^2+1?

Solution by the *Editors*.

Any eight-digit square of the proposed form may be written

$$N^2=(aB^5+bB^4+cB+d)(B^2+1).$$

If B^2+1 is not divisible by a square factor, N^2 must be divisible by $(B^2+1)^2$ and we have, for some integer K,

$$aB^5+bB^4+cB+d=K^2(B^2+1). \tag{1}$$

But the left member is less than B^6, whence K^2B^2 is less than B^6. Since thus K is less than B^2, it must be of the form $K=mB+n$ and therefore

$$N=K(B^2+1)=mnmn \tag{2}$$

as required.

By division (1) may be put in the form

$$K^2=aB^3+bB^2-aB-b+[(a+c)B+(b+d]/(B^2+1). \tag{3}$$

Since the last term is an integer and its numerator is positive but less than $2B^2$, it must have the value unity, or

$$(a+c)B+(b+d)=B^2+1. \tag{4}$$

The equation (3) can then be written in the convenient form

$$(aB+b)(B^2-1)=K^2-1, \tag{5}$$

which will be satisfied (with appropriate a,b) by every value of K such that

$$B<K<B^2, \quad K^2\equiv 1 \pmod{B^2-1}. \tag{6}$$

Choosing $K=B^2-B-1$ and $K=B^2-2$, which satisfy (6), we obtain from (2), (4) and (5) the following values for $N^2=ababcdcd$,

$$\text{(7)}\quad \overline{B-2}\ 0\ \overline{B-2}\ 0\ 2\ 1\ 2\ 1=(\overline{B-2}\ \overline{B-1}\ \overline{B-2}\ \overline{B-1})^2,$$

$$\overline{B-1}\ \overline{B-3}\ \overline{B-1}\ \overline{B-3}\ 0\ 4\ 0\ 4=(\overline{B-1}\ \overline{B-2}\ \overline{B-1}\ \overline{B-2})^2,$$

valid for any B greater than 4. These results are also valid for $B=3$ and $B=4$ provided the last four digits of the second square are written 1111 and 1010 respectively.

Other solutions of (6) are possible if B^2-1 is divisible by more than two primes or is even (and not 8). Thus for example if $B\equiv 1 \pmod 4$ we find $ababcdcd=(mnmn)^2$ for

$$a=(B-1)/4,\quad b=a+1,\quad c=3a,\quad d=c+1,\quad m=2a,\quad n=2a+1;$$

$$a=(B-1)/4,\quad b=a-1,\quad c=3a,\quad d=c+3,\quad m=2a,\quad n=2a-1,$$

in which if $B=5$, read $c=4$, $d=1$. For $B\equiv 3 \pmod 4$ and $B\neq 3$, we have

$$a=(B-3)/4,\quad b=3a+3,\quad c=b-1,\quad d=a+1,\quad m=2a+1,\quad n=2a+2;$$

$$a=(B-3)/4,\quad b=3a+1,\quad c=b+1,\quad d=a+3,\quad m=2a+1,\quad n=2a.$$

These and the results in (7) are valid regardless of the factors of B^2+1. Thus for every value of B, except 2, there are at least two squares of the proposed form. If, however, B^2+1 is divisible by a square, then the argument leading to (2) and (5) is no longer the only possibility and we may have yet other squares $ababcdcd$ which are not of the form $(mnmn)^2$. For example with base 7, $12123232=(3014)^2$.

No. 463. Proposed by *E. P. Starke*, Rutgers University.

Prove the construction for the tangent to any conic at one of its points P: join P to a focus F; let the perpendicular to PF at F meet the corresponding directrix at D; DP is the required tangent.

I. Solution by *D. L. MacKay*, Evander Childs High School, New York City.

Let P and P' be two neighboring points on the conic, and let E and G be their projections on the directrix. Let PP' meet the directrix in D', and let PF cut the conic again in H. By definition, $FP:PE=FP':P'G$, and from similar triangles, $PE:P'G=PD':P'D'$. Hence we have $FP:FP'=PE:P'G=PD':P'D'$, so that $D'F$ bisects angle $P'FH$. Now as P' approaches P along the curve, the angles $D'FP'$ and $D'FH$, being always equal, become right angles when FP' coincides with FP and $D'P$ becomes the tangent DP. The relationship upon which the construction depends is thus established.

II. Solution by *Paul D. Thomas*, Lucedale, Miss.

The equation of a conic of eccentricity e having the Y-axis as a directrix and $F(a,0)$ as the corresponding focus is

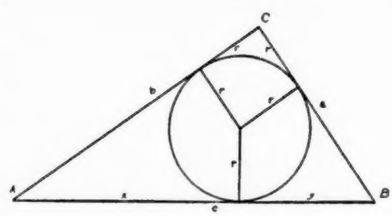
$$x^2(e^2-1)+2ax-y^2-a^2=0.$$

The equation of the tangent to the conic at $P(x_1,y_1)$ is

$$x_1x(e^2-1)+a(x+x_1)-y_1y-a^2=0.$$

This line meets the Y-axis in the point $D(0,\ a(x_1-a)/y_1)$. Thus the slope of FD is $-(x_1-a)/y_1$, while that of PF is $y_1/(x_1-a)$. Since these slopes are negative reciprocal, FD is perpendicular to PF at F.

A Proof of the Theorem of Pythagoras, by *Guy E. Clutter,* Houston, Texas.



The area of ABC is (r = in-radius):

(1) $$ab/2=r(a+b+c)/2$$

in which $$2(r+x+y)=a+b+c$$ or

(2) $$2r=a+b-c.$$

The elimination of r between (1) and (2) produces

$$2ab=(a+b-c)(a+b+c)$$

or $$a^2+b^2=c^2.$$

PROPOSALS

No. 486. Proposed by *V. Thébault,* Tennie, Sarthe, France.

In which system of numeration having a base less than 10000 are there the largest number of 4-digit squares of the form *aabb* and such that $aabb=(cc)^2$?

No. 487. Proposed by *W. N. Huff,* The Hill School, Pottstown, Pa.

Find the volume generated by revolving a cube about a diagonal.

No. 488. Proposed by *John Bristow,* student, Colgate University.

Two parabolas with axes perpendicular pass through the point P, (2,8), and have there the same tangent line and the same circle of

curvature. If the equation of one of them is $y=6x-x^2$, find the equation of the other.

No. 489. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Given the quadrilateral $ABCD$. Find two points X and Y on AB and CD, respectively so that AX, XY, and YC are proportional to m, n, p.

No. 490. Proposed by *Howard D. Grossman*, New York City.

A chance event which can be either A or non-A is repeated n times. Let E be the compound event of exactly r and no more A's. Uspensky (*Introduction to Mathematical Probability*, pp. 77-84) has given a treatment of the problem: find the probability that the sequence of n events will contain at least one E. The following problem is simpler and in many cases more significant: find the probable number of E's and show that it is asymptotic to $p^r(1-p)^2n$, where p is the probability of the occurrence of A.

No. 491. Proposed by *Franklin Miller, Jr.*, Rutgers University.

A light stands at A within the angle formed by two intersecting mirrors. Show that the length of the path of a ray of light which returns to A after being reflected in both mirrors equals twice the distance between A and the intersection of the mirrors times the sine of the angle between them.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Mathematical Monographs, Volume I (Northwestern University Studies in Mathematics and Physical Sciences, No. 1). Northwestern University, Evanston and Chicago, 1941. vii – 172 pages.

The four monographs in this volume are outgrowths of four sets of lectures in the field of analysis and its applications, delivered during the year 1939-40 at Northwestern University. The titles of the monographs are: Maxima and minima of functions of two or more variables; The statistics of time series; Topics in continued fractions and summability; Spectra in quadratic forms in infinitely many variables. The respective authors are D. R. Curtiss; H. T. Davis; H. L. Garabedian and H. S. Wall; E. D. Hellinger. The volume was prepared under the editorial supervision of D. R. Curtiss.

Monographs of this type fulfil a very useful role in enabling the student and research worker in the field of mathematics to rapidly acquaint himself with recent developments in certain lines of mathematical advance, not yet available in general treatises and difficult of access in the memoirs of research journals. Those in the present volume have been well prepared from this point of view; they were written by well known experts in the fields in question and contain some of the original work of these men. In each case, however, enough general background has been given to make an understanding of the discussion within the scope of a reader well grounded in analysis.

In the first monograph the novel features of the discussion center about tests for maxima and minima which have analogies in the theory of equations. In particular there is an interesting extension of the Sturmian algorithm to functions of two variables, from which criteria for a proper extreme of a function are derived.

In the second monograph there is given a systematic discussion of the statistical theory of time series as contrasted with the statistics of frequency distributions. Applications of the theory to problems of a dynamic nature in the field of economics are included.

In the third monograph recent work of the writers which establishes the connection between continued fractions and Hausdorff methods of summability is set forth in a systematic exposition. This development completes earlier work of Stieltjes and Hausdorff in which the connection between the moment problem and continued fractions and that between the moment problem and summability theory were respectively developed.

The last monograph deals with the theory of quadratic forms in infinitely many variables, as developed by Hilbert and his school. It is an excellent introduction to this important field, as one would expect from one of the leading contributors to the theory. In order to avoid complexity the author confines himself to Hilbert's original space of a countably infinite number of variables instead of developing the subject from a more general standpoint. This has the advantage of making the discussion more accessible to the average reader.

University of Cincinnati. CHARLES N. MOORE.

Spherical Trigonometry and Tables. By Granville, Smith, Mikesh. Ginn and Company, Boston, 1942. xxii+103 pages. $1.25.

This book is that part of the larger Granville's *Plane and Spherical Trigonometry and Tables* which deals with spherical trigonometry, with the addition of a résumé of the formulas of plane trigonometry and an introduction to the terminology of navigation. Chapters I, II contain the theory of right and oblique spherical triangles, respectively, with the usual memory devices of Napier. Chapter III gives several applications to the celestial sphere. Chapter IV summarizes the results of the first two chapters. The tables (four-place) are five in number: *logarithms of numbers, logarithms of trigonometric functions* (degrees and minutes, also degrees and decimal parts of a degree), natural values of *sine, cosine, tangent* and *cotangent.*

In these days of revived interest in the subject of spherical trigonometry, it is convenient to have separate books complete in themselves for all who have had plane trigonometry previously. This volume meets such a need and also provides a varied collection of problems.

Virginia Military Institute. W. E. BYRNE.

Mathematics Dictionary. By Glenn James, assisted by R. C. James. The Digest Press, Van Nuys, California, 1942. v+259, and 22 additional pages of tables and lists of symbols.

Because of the individual abilities and weaknesses of their authors, because also of the individual character and domains of their contents, dictionaries differ widely in the nature of their appeal to users. Some are consulted briefly and matter-of-factly during hours of work; others serve better for browsing during hours of leisure. This review makes the prediction that the readers of this mathematics dictionary will rarely come to consult, and, when they do, will stay to browse. It is the view of this review that it is no discredit to a dictionary to be a meadow instead of a mine, that it is no discredit to a dictionary to be a source of diversion rather than a source of information.

Reviewers tend to form the habit of estimating only the achievement of the author's avowed objectives, and, by thus viewing books only from their best sides, to give readers a narrower glimpse of their imperfections. This remark is intended here neither as praise nor as criticism of reviewers, but merely as an introduction of the avowed objective of the authors above. This is, in their words on page iv, "the conveying of mathematical concepts. Formalism has throughout been a minor consideration. The needs of three classes of readers have been kept constantly in mind: (1) The student's need of a reliable supplement to his current work and his need for ready access in the more advanced classes to the multitude of facts he has already studied. (2) The needs of teachers and workers in other fields that require mathematics. (3) The needs of the intelligent layman who wants to know how to answer the mathematical questions that confront him almost daily."

The actual book, however, seems to have been shaped not so much by the motive which the authors confess as by the means which they have used to realize it. They describe this in their own words as follows. "For the most part this dictionary is a reporting from modern textbooks." A reporting is truly what is looks like.

Upon these modern texts, and not upon the authors of the dictionary, may consequently be laid the blame for the "ragged upper frontier of the collection of definitions". These words of the authors seem apt enough for a collection which includes moment of inertia but not product of inertia, group but not ring, logistic but not logic, summation but not summand, ideal point but not ideal number, Fourier series but not orthog-

onal series, ordered set and dense set but not open set and not closed set. They aptly characterize also such irregularities as listing double cusp, double integral, double root, and double subscript under *double*; but not double series, which has to be content with hiding out under series; and explaining ruler in terms of straight edge, without explaining straight edge: but the blame for these is not so easy to assign. And when it comes to a possibly misleading definition (see the definition of limit point of a set), then it could be hoped that the authors of the dictionary had been more critical, in a fatherly way, of their textbook sources.

There are, furthermore, some defects which must be attributed wholly to the authors. Among these are leading words in subheading type—for instance (1) Menelaus' Theorem on page 153, and (2) Rules for use of Logarithms on page 210—and the use in the text of symbols not explained in the glossary of symbols at the back of the book—for instance the symbol (a,b) for straight line interval or segment, whichever it is, from a to b occurring under Rolle's Theorem on page 208. There is also some bad type which ought not to have appeared, conspicuously the symbol omega on page 142.

It is perhaps not fitting to keep on pointing out small faults in a project which has already required more than three thousand hours of work and which its authors do not hesitate to defend by quoting some celebrated sentences of Samuel Johnson. Furthermore the contents really have surprising scope in a short space, the scope extending from five lucid lines upon *point* to an impenetrable column and a half on *tensor*. An appreciation of this impressive range may be gotten from a glance at the proper names which it includes. Among these are Newton's Laws of Motion, Gauss' Proof, Leibniz's Theorem, Euler's Constant, Bessel Functions, Legendre's Polynomials, Lagrange's Method, Green's Theorem, Peaucellier's Cell, and the Cauchy-Lipschitz Condition. At the end of such a list as this it dawns upon one that the great names of mathematics to this dictionary are merely adjectives. Newton is not, for instance, a man who invented the differential calculus, but merely a collaborator with Gregory (another adjective) in an interpolation formula, a qualifier of some laws of motion, and a kind of approximation formula. Descartes is merely a variety of folium and a particular rule of signs, thus having a purely coincidental connection with geometry. It may be said further that this dictionary defines terms and does not deal with biographies. But when a strict adherence to any program makes Archimedes the mere modifier of a spiral, reduces Pythagoras to the actual adjective Pythagorean, and omits both Bolyai and Lobachevski completely, then a reader with a view of mathematics as human in addition to divine comes to wish for a few good stiff derelictions.

As to the great logicians, they do not rate at all. Not a one of them from Aristotle to Goedel is mentioned, even as an appendage to some gimcrack. In fact the dictionary, although positively devoted to mathematics of investment, flattering to physics, and aware of astronomy, ignores logic completely. Not even such words as syllogism and proposition are listed. Is it implied that the needs of students, teachers, and intelligent laymen never include logic?

Some of the space saved on history and logic in the book is used in giving examples and listing synonyms. The examples are almost certain to be useful. Some synonmys have been overlooked, for instance *complanar*. Some of the rest of the space has been spent, less defensibly, on oddities. It seems almost as if the authors had at times acted out of whimsey, picking up strays and listing them kindly to preserve the neglected things from oblivion. Are not astroid, versiera, round angle, and the Dayton Ohio Plan examples? But it is just such inclusions as these which make the book a good field in which to browse. And that, to repeat, is what this reviewer maintains it primarily is; in spite of loftier motives and needs one, two, and three in the minds of its authors.

Baton Rouge, La. N. E. RUTT.

Logarithms, Trigonometry, Statistics. By Cooley, Graham, John, and Tilley. McGraw-Hill Book Company, New York, 1942. xii+280 pages. $2.00.

This textbook presents the standard material of plane trigonometry and logarithms, together with some of the descriptive methodology of statistics. Chapter titles are as follows: I, *Logarthms.* II, *The Slide Rule.* III, *Errors in Numerical Calculation.* IV, *Logarithmic and Exponential Functions.* V, *Indirect Measurement* (*Right Triangles*). VI, *Angles of Any Size and Their Functions.* VII, *Oblique Triangles.* VIII, *Trigonometric Analysis.* IX, *Law of Tangents and the Half-Angle Law.* X, *Trigonometric Functional Relations.* XI, *Inverse Trigonometric Functions.* XII, *Polar Coordinates.* XIII, *Polar Representation of Complex Numbers.* XIV, *Empirical Functions.* XV, *Statistical Functions.* Chapters II and III, and XII to XV inclusive, are starred for possible omission. Six sections among the other chapters are similarly starred.

Trigonometry has been referred to somewhere as "that homely, perhaps, but most serviceable handmaid to so many of the arts and sciences". The styles in which authors and publishers dress this humble handmaiden seem almost as varied as the fashions of milady's wardrobe. To be sure, each season brings catalogs from publishers announcing new styles in that general field which Hardy, in his *Apology,* calls "school mathematics". But it is probably trigonometry that leads the field. There are the fat, or plump, designs which feature "full and complete treatments", and there are the streamlined models in "twenty lessons". Critics of the former have accused the authors of confusing simplification with amplification, claiming that the amplification has spent itself upon details rather than upon principles. The latter model has been both praised and criticized, on the one hand for being "commonly brief" and, on the other, for being "cut to the bone—or deeper". Lately, "essentials with applications" seem to be the mode. Some of these may have been manufactured, one suspects, with an eye to the market opened by the war.

The volume under review appears to set a new fashion with respect to titles. Although the title is an unusual one, it is not so fantastic as it might seem at first sight. This book, together with a companion volume on college algebra, is intended to serve for a course in first year college mathematics, the emphasis here being upon those functional relations which might be called "non-algebraic". On the jacket we find the following description of the book: "... Slide rule and growth problems are dealt with in connection with logarithms. Trigonometry is applied to vibratory motion, polar coordinates, and complex numbers. Coordinate geometry of transcendental functions is discussed, and there are chapters on empirical functions and statistics." This turns out to be an accurate thumbnail sketch, but a few supplementary comments may be welcomed. The trigonometric functions are first defined for the right triangle. Use of the unit mil in angular measurement is explained (in a starred section). The laws of sines and of cosines are proved without use of projections. The addition formulas for sine and cosine functions are established for both the restricted and unrestricted cases. (Incidentally, the reviewer would like to see in some textbook on trigonometry the proofs that appeared in the *American Mathematical Monthly,* Vol. 49, pages 325 and 327, for the addition and half-angle formulas.) In the chapter on empirical functions, methods are given for fitting the straight line, power function, and exponential function. A brief discussion of finite differences is included as a setting for the Gregory-Newton formula and the use of this formula in fitting polynomials is explained. The chapter on statistics deals mainly with methods of computing averages and measures of dispersion in descriptive characterizations of frequency distributions. Here the standard deviation is defined in accordance with the definition which is usually given in

first courses, not the formulation which defines this statistic as an unbiased estimate of the corresponding parameter in the underlying probability distribution.

There are no tables in the book. Instead, the student is referred to the Mathematical Tables in the *Handbook of Chemistry and Physics* (published by the Chemical Rubber Company, Cleveland, Ohio). Lists of problems seem to be liberal, anyhow adequate, and answers are given to a good many of them. The text is attractively printed in 11 point type on machine coated paper.

University of Wisconsin. J. F. KENNEY.

Aircraft Mathematics. By S. A. Walling and J. C. Hill. The Macmillan Co., New York, 1942. 189 pages.

This small English book was designed primarily as a progressive course for the Air Training Corps Cadets. In its construction the aim was two-fold: first, to afford a rapid review of the elementary processes; secondly, to show the application of these processes in the various Services.

The subject-matter is divided into six sections (there being no chapters) which are entitled, in order: *elementary arithmetic, elementary algebra, graphs, geometry, logarithms* and *trigonometry.* Four-place tables of logarithms and answers are given, but there is no index.

In each section the reviews are concise, the topics considered usually being limited to those which will be used in the problems to follow. Rules of procedure are in most cases merely stated, no attempt being made at the rationalization which would make for a deeper understanding of the processes involved. However, the authors expressly state that the book does not pretend to be a textbook of mathematics.

The wealth of timely applied problems, introduced by the well written technical information necessary for their solution, gives promise of creating for the student an "enlivening and instructive interest in the special branches of the Services," which, in turn, should lead him to an appreciation of the value of a sound basic training in elementary mathematics.

The reviewer does not approve the use of $+ve$ and $-ve$ for positive and negative (p. 62), and prefers not to use both *cotan* and *cot* (p. 153). These are but minor criticisms of an excellent book that has evidently been well received in Great Britain, where two editions and a reprint (with corrections) have been published within the ten months prior to the publication of the American Edition.

This book is worthy of the attention of college students of mathematics, particularly those who are interested in aerial navigation.

Northern Illinois State Teachers College. NORMA STELFORD.

Trigonometry, Plane and Spherical. By Miles C. Hartley. The Odyssey Press, New York, 1942. 298 pages; $1.60.

This book is very complete, and the explanations are very clear. The introductory chapter, giving a brief summary of the important elementary principles of algebra and analytic geometry, will be welcomed by many students and teachers. There is an ample collection of exercises and worked illustrative examples. While the four-place logarithmic and trigonometric tables are probably sufficient for practical applications, it seems probable that some students will use five-place tables found in many other

texts, thus avoiding practice in interpolation. The summaries, tests, and review exercises are very helpful to students.

The misprint

$$\text{“}\tan\alpha = \frac{\text{opposite side}}{\text{hypotenuse}}\text{”}$$

near the middle of page 80 is very unfortunate. The proofs of identities on pages 103 and 104 illustrate a type of error very common among students, who start with the identity which is to proved and transform it into an obvious identity. The following, in the reviewer's opinion, is a sound proof of illustrative example 1:

$$\frac{\cot\Theta}{\csc\Theta} = \frac{\left(\dfrac{\cos\Theta}{\sin\Theta}\right)}{\left(\dfrac{1}{\sin\Theta}\right)} = \cos\Theta.$$

(The second step is to multiply numerator and denominator by $\sin\Theta$). In §109, the plane BED is assumed, but not stated, to be perpendicular to the line OA. In §116 there seems to be confusion of notation. If the letters B and C are interchanged in the drawings and in the first three lines of the section, the discussion will be correct. The general appearance of the book is attractive and it will undoubtedly find wide use.

Southwestern University (Georgetown, Texas). HORACE L. OLSON.